

# Giant amplification of modes in parity-time symmetric waveguides

Vladimir V. Konotop<sup>1</sup>, Valery S. Shchesnovich<sup>2</sup>, and Dmitry A. Zezyulin<sup>1,\*</sup>

<sup>1</sup>*Centro de Física Teórica e Computacional, and Departamento de Física, Faculdade de Ciências, Universidade de Lisboa, Avenida Professor Gama Pinto 2, Lisboa 1649-003, Portugal*

<sup>2</sup>*Centro de Ciências Naturais e Humanas, Universidade Federal do ABC, Santo André, São Paulo 09210-170, Brazil*

\*Corresponding author: zezyulin@cii.fc.ul.pt

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## Abstract

The combination of the interference with the amplification of modes in a waveguide with gain and losses can result in a giant amplification of the propagating beam, which propagates without distortion of its average amplitude. An increase of the gain-loss gradient by only a few times results in a magnification of the beam by a several orders of magnitude.

**Keywords:** Optical waveguides, Parity-time symmetry, Amplification

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## 1. Introduction

Amplification of guided optical waves is necessary to compensate for unavoidable losses along the propagations distance. The amplification is also important in the initiation process for the pulse generation requiring high intensities with, possibly, very low intensity inputs. Additionally, the concept of the gain-guidance, where the optical fibers and planar waveguides with gain allow for the efficient propagation of low power modes, was suggested recently [1] as an alternative to the conventional index-guiding structures. The phenomenon was observed in the Nd-doped optical fibers with low refractive index profiles [2] and served as a basic element for the single-mode optical laser [3]. Later on, the idea of gain-guidance was extended also to the nonlinear media [4, 5].

The gain element causes either the unbounded growth of the signal intensity or, if nonlinear dissipation becomes dominant, the convergence to some stationary or pulsating solution (i.e. to an attractor) [6]. In all cases the behavior of the system is fully determined by the values of the system parameters and does not depend on the characteristics of the input signal (provided that the input signal belongs to the attractor basin). Therefore, in the linear systems one usually considers distortions of modes due to the gain (implemented, say, by doping), rather than propagation of such modes for long distances.

On the other hand, a large wave intensity in some spatial domains can be achieved by the constructive interference of modes. In such amplification scenario, the increase of the intensity of the superimposed modes is determined solely by the characteristics of the input beam, and as such is limited by an *a priori* given value.

In this Letter we propose to exploit *the interference of the gain-guided modes* which, on the one hand, allows for a flexible control of the output intensity by variation of

the input amplitude, and, on the other hand, benefit from the amplifying properties of the medium, thus resulting in a giant amplification of input beams over relatively short propagation distances. Moreover, by a proper design of the dissipation and gain profiles it is possible to achieve the guidance of modes for an arbitrary distance, without overcoming an *a priori* given intensity limit, and to realize either single or multichannel guidance (selected by changing the input beam) inside the same waveguide without changing the waveguide properties. Below, we focus on the linear case, when the effects of nonlinearity can be neglected.

Our idea is based on the use of the special design for the gain and loss distribution to assure the parity-time (or  $\mathcal{PT}$ ) symmetry of the system. Suggested in the quantum mechanical context [7], this concept implies spatial ( $\mathcal{P}$ ) and temporal ( $\mathcal{T}$ ) symmetry of the system, which in optics this can be implemented by means of even distribution of the real part of the refractive index of the guiding medium combined with the anti-symmetrically distributed gain and losses [8]. Particular relevance of the  $\mathcal{PT}$ -symmetric media for optical applications is justified by the property that in a definite range of parameters of such media all linear modes have purely real propagation constants (which corresponds to the pure real spectrum of the quantum-mechanical Hamiltonian [7]). In other words, this means that *all* linear modes propagate without infinite growth or decay, thus allowing for the mode guidance in media with gain and losses. It is relevant to notice here that the  $\mathcal{PT}$  symmetry may be not sufficient for existence of purely real spectrum. The situation when a  $\mathcal{PT}$ -symmetric system acquires (for some values of the parameters) complex eigenvalues (propagation constants) is usually referred to as spontaneous  $\mathcal{PT}$ -symmetry breaking. In this Letter, however, we consider a situation when  $\mathcal{PT}$  symmetry breaking does not occur.

The setup for the experimental implementation of  $\mathcal{PT}$ -symmetric systems with gain and losses using the wave guiding structures was proposed in [8], and later developed in numerous studies (e.g., for the linear wave propagation see [9, 10]). The first experimental studies of the optical  $\mathcal{PT}$ -symmetric structures were reported in [11, 12]. In particular, such phenomena as spontaneous  $\mathcal{PT}$  symmetry breaking and power oscillations violating left-right symmetry, have been experimentally observed. It was also suggested that appropriately designed combinations of the gain and losses in a dual core coupler can be used for the amplification of light signals [13] and for the noise filtering [14].

## 2. Some properties of $\mathcal{PT}$ -symmetric potentials

We consider propagation of a paraxial beam governed by the dimensionless field  $q$  obeying the Schrödinger equation

$$iq_z = \mathcal{H}q, \quad \mathcal{H} = -\nabla^2 + V(\mathbf{r}) + iW(\mathbf{r}), \quad (1)$$

where  $\mathbf{r} = (x, y)$ ,  $\nabla = (\partial_x, \partial_y)$ ,  $r = |\mathbf{r}|$ ,  $V(\mathbf{r})$  describes radially symmetric modulation of the waveguide refractive index, while  $W(\mathbf{r})$  describes a spatial distribution of the gain and losses. We focus on the case

$$V(-\mathbf{r}) = V(\mathbf{r}) \quad \text{and} \quad W(\mathbf{r}) = -W(-\mathbf{r}), \quad (2)$$

which is a necessary condition for the operator  $\mathcal{H}$  to be  $\mathcal{PT}$  symmetric. In particular, we choose [15]

$$V(r) = r^2, \quad W(\mathbf{r}) = 2g_1x + 2g_2y = 2\mathbf{g} \cdot \mathbf{r} \quad (3)$$

where  $\mathbf{g} = (g_1, g_2) = \frac{1}{2}\nabla W$  characterizes the gradient of the gain-loss term having the amplitude  $g$ :  $g^2 = g_1^2 + g_2^2$ . It is relevant to notice that the beam amplification reported below, although less pronounced because of the interference of a smaller number of modes, can also be observed in the one-dimensional setting. The two-dimensional setting, besides giving a considerably stronger amplification, offers some additional effects, such as the beam splitting, also considered below.

For the stationary modes  $q(z, \mathbf{r}) = e^{i\beta_{\mathbf{n}}z}\phi_{\mathbf{n}}(\mathbf{r})$ , we obtain the eigenvalue problem  $\mathcal{H}\phi_{\mathbf{n}} = -\beta_{\mathbf{n}}\phi_{\mathbf{n}}$ , where  $\mathbf{n} = (n_1, n_2)$  is an ordered pair of nonnegative integers identifying the guided modes. This problem has a pure real spectrum [16]:  $\beta_{\mathbf{n}} = -(2n_1 + 2n_2 + 2 + g^2)$  with the respective eigenfunctions given by

$$\phi_{\mathbf{n}}(\mathbf{r}) = \frac{H_{n_1}(x + ig_1)H_{n_2}(y + ig_2)}{\sqrt{2^{n_1+n_2}n_1!n_2!\pi}} \times e^{-\frac{1}{2}[(x+ig_1)^2 + (y+ig_2)^2]}, \quad (4)$$

where  $H_n(\xi)$  are the Hermite polynomials. Since all  $\beta_{\mathbf{n}}$  are real, all the linear eigenmodes propagate undistorted.

In addition,  $\tilde{\phi}_{\mathbf{n}} = \phi_{\mathbf{n}}^*$  (the asterisk stands for the complex conjugation) is an eigenvector of the Hermitian conjugated operator  $\mathcal{H}^\dagger$ :  $\mathcal{H}^\dagger\tilde{\phi}_{\mathbf{n}} = -\beta_{\mathbf{n}}\tilde{\phi}_{\mathbf{n}}$ . The two sets,  $\tilde{\phi}_{\mathbf{n}}$  and  $\phi_{\mathbf{n}}$ , constitute the left and right complete bases [16].

The left and right eigenvectors are mutually orthogonal and normalized:

$$(\tilde{\phi}_{\mathbf{n}}, \phi_{\mathbf{m}}) = \int \phi_{\mathbf{n}}(\mathbf{r})\phi_{\mathbf{m}}(\mathbf{r})d\mathbf{r} = \delta_{n_1, m_1}\delta_{n_2, m_2}, \quad (5)$$

where  $\delta_{n, m}$  is the Kronecker delta. Therefore, for a given input  $q_0(\mathbf{r}) = q(0, \mathbf{r})$ , the field evolution can be found in the form

$$q(z, \mathbf{r}) = \sum_{\mathbf{n}} c_{\mathbf{n}} e^{i\beta_{\mathbf{n}}z} \phi_{\mathbf{n}}(\mathbf{r}), \quad c_{\mathbf{n}} = (\phi_{\mathbf{n}}^*, q_0). \quad (6)$$

However, the right eigenmodes  $\phi_{\mathbf{n}}(\mathbf{r})$  [as well as the left ones  $\tilde{\phi}_{\mathbf{n}}(\mathbf{r})$ ] are not orthogonal in the usual sense. Their scalar product reads

$$(\phi_{\mathbf{n}}, \phi_{\mathbf{m}}) = \int \phi_{\mathbf{n}}^*(\mathbf{r})\phi_{\mathbf{m}}(\mathbf{r})d\mathbf{r} = D_{n_1 m_1}(g_1)D_{n_2 m_2}(g_2), \quad (7)$$

which means that the optical energy distribution between the modes changes during the propagation. The energy transfer between the  $\mathbf{n}$ -th and  $\mathbf{m}$ -th eigenmodes is described by the functions

$$D_{nm}(g_j) = (\pm i g_j)^{n+m-2\mu} 2^{\frac{\kappa}{2}} e^{g_j^2} \left( \frac{n!}{m!} \right)^{\pm \frac{1}{2}} L_{\mu}^{(\kappa)}(-2g_j^2), \quad (8)$$

where  $L_{\mu}^{(\kappa)}(\xi)$  are the generalized Laguerre polynomials,  $L_{\mu}^{(0)}(\xi) \equiv L_{\mu}(\xi)$ ,  $\kappa = |n - m|$ ,  $\mu = \min(n, m)$ , and the signs “+” and “−” stand for the cases  $m \geq n$  and  $m < n$ , respectively. Since the Laguerre polynomials do not possess negative roots [17], the coefficients  $D_{nm}(g_j) \neq 0$ , unless  $g_j = 0$ . Since the spectrum  $\beta_{\mathbf{n}}$  is equidistant, we conclude from (6) that the solution  $q(z, \mathbf{r})$  as well as the energy flow  $U(z) = \int |q|^2 d\mathbf{r}$  are  $\pi$ -periodic with respect to  $z$ . We emphasize that though the equidistant spectrum simplifies considerably the analysis, it is not necessary for a giant amplification induced by the interference of the gain-guided modes, which can be observed also for other profiles of the refractive index. Notice also that energy oscillations are known to be a typical feature of  $\mathcal{PT}$ -symmetric systems [10, 12].

We intend to explore the fact that, the modes which in the “conservative” (i.e. index guiding) waveguides have zero intensity at the center of the waveguide, for  $g^2 \neq 0$  carry nonzero power, and moreover, for a sufficiently large  $g^2$  the respective modes of the  $\mathcal{PT}$ -symmetric operator  $\mathcal{H}$  may acquire the global maximum precisely at  $\mathbf{r} = 0$ . Moreover, we show that in the case of a  $\mathcal{PT}$ -symmetric profile with a sufficiently large  $g^2$ , the intensity maximum is shifted to the mode with a high quantum number  $\mathbf{n}$ , i.e. with a large value of the propagation constant. This feature leads to a dramatically different interference pattern for such modes, allowing for a constructive interference of the maxima of many different modes precisely at  $\mathbf{r} = 0$ , thus strongly enhancing the intensity of the beam. At the same time, a destructive interference of the modes outside the beam center may fully eliminate all of the secondary interference fringes observable in the conventional patterns.

### 3. Giant amplification

Let us consider the input beam of a Gaussian shape, which is the lowest eigenmode of a conservative waveguide with the same refractive index [i.e. of (1)–(3) with  $\mathbf{g} = 0$ ]:  $q_0(r) = \sqrt{U_0/\pi} e^{-r^2/2}$ , where  $U_0 = U(0) = \int |q_0|^2 d\mathbf{r}$  is the input beam energy flow. A remarkable fact is that the respective solution  $q(z, \mathbf{r})$  can be found in an explicit form:

$$q = \sqrt{\frac{U_0}{\pi}} e^{-i(2+g^2)z - 2i\mathbf{g} \cdot \mathbf{r} \sin^2 z - ig^2 [\sin(4z)/4 - \sin(2z)]} \times e^{2g^2 \sin^2 z} e^{-\frac{1}{2}[(x-g_1 \sin(2z))^2 + (y-g_2 \sin(2z))^2]}. \quad (9)$$

Moreover, formula (9) is a particular case of a more general family of explicit solutions to (1)–(3) given by [18]:

$$q(z, \mathbf{r}) = \sqrt{U_0} e^{i\beta_{\mathbf{n}} z + i\mathbf{a} \cdot \mathbf{r} \cos(2z) - i\mathbf{a} \cdot \mathbf{a} \sin(4z)/4} \times e^{2\mathbf{a} \cdot \mathbf{g} \sin^2 z - g^2/2} \frac{\phi_{\mathbf{n}}(x - X_1(z), y - X_2(z))}{\sqrt{L_{n_1}(-2g_1^2)L_{n_2}(-2g_2^2)}}, \quad (10)$$

where  $\mathbf{a} = (a_1, a_2)$  is an arbitrary vector, which controls the angle of the incidence of the input beam, and  $X_{1,2}(z) = a_{1,2} \sin(2z)$  determine the position of the center of the mode. Indeed, (9) is obtained from (10) simply by setting  $\mathbf{a} = \mathbf{g}$  and  $\mathbf{n} = (0, 0)$ . The energy of the beam described by (10) reads  $U(z) = U_0 e^{4\mathbf{a} \cdot \mathbf{g} \sin^2 z}$ . In accordance with the expansion (6),  $U(z)$  is a  $\pi$ -periodic function. If the product  $\mathbf{a} \cdot \mathbf{g}$  is positive, i.e. the beam is tilted to penetrate deeper in the domain with the gain, then  $U(z)$  approaches its maxima  $U_{max} = U_0 e^{4\mathbf{a} \cdot \mathbf{g}}$  at  $z_k = \frac{\pi}{2} + k\pi$  and there is an exponential amplification of the input amplitude achieved due to a combined effect of the gain-loss gradient  $\mathbf{g}$  and a properly chosen direction of the input beam, i.e.  $\mathbf{a}$ . Negative  $\mathbf{a} \cdot \mathbf{g}$ , on the other hand, results in a periodic attenuation of the beam.

Returning to expression (9) we find that the amplitude of the solution has the Gaussian shape with the center oscillating along the direction indicated by  $\mathbf{g}$ . For  $0 < z < \pi/2$  the center of the beam is situated in the domain of gain, and the energy  $U(z)$  grows. The maximal amplification is achieved in the point  $z = \pi/2$ , i.e. when the center of the beam returns to the origin (where the gain is compensated by losses). For  $\pi/2 < z < \pi$  the beam resides in the lossy domain and loses the energy. At  $z = \pi$  it returns to the origin and its energy approaches the lowest value  $U_0$ . Then the process is repeated (this recursive behavior is a consequence of the  $\mathcal{PT}$  symmetry and resembles dynamics in a Hamiltonian system).

To illustrate the above findings we focus, on the case of  $\mathbf{g} = (g_1, 0)$ . At first, we consider the particular case given by (9). The beam energy flow  $U(z)$  and the beam profiles at the maximal amplification point are shown in Fig. 1 (a)–(c). Figure 1 reveals several important features. First, even a weak increase of the gradient results in a giant amplification of the beam intensity as it is seen from panels (b) and (c) of Fig. 1. Whereas these two panels are obtained for the same Gaussian input beam, an increase

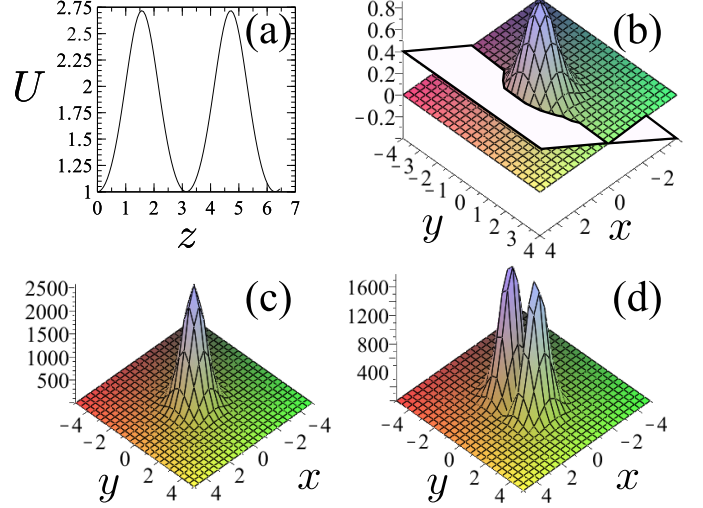


Figure 1: (a) Energy flow *vs* propagation distance and (b) profile  $|q(\pi/2, \mathbf{r})|^2$  for the solution (9) with gradient  $\mathbf{g} = (1/2, 0)$ . In the panel (b) the gain landscape  $W(\mathbf{r})$  for  $g_1 > 0$  and  $g_2 = 0$  is schematically shown. (c) Profile  $|q(\pi/2, \mathbf{r})|^2$  for the solution (9) with  $\mathbf{g} = (3/2, 0)$ . (d) Profile  $|q(\pi/2, \mathbf{r})|^2$  of the solution (10) with  $\mathbf{n} = (0, 1)$  and gradient  $\mathbf{g} = \mathbf{a} = (3/2, 0)$ . In all panels  $U_0 = 1$ .

Table 1: Absolute values of the coefficients  $c_{(n_1, 0)}$  for  $U_0 = 1$ ,  $g_2 = 0$  and different  $g_1$  [for  $n_2 \neq 0$  all  $c_{(n_1, n_2)} = 0$ ].

$n_1$	0	1	2	3	4
$g_1 = 1/2$	1.06	0.37	0.09	0.02	0.003
$g_1 = 3/2$	1.76	1.86	1.40	0.85	0.45

of the gradient amplitude  $g$  by three times [from  $g = 1/2$  in panel (b) to  $g = 3/2$  in panel (c)] results in magnification of the peak intensity by about  $3 \times 10^3$  times [the peak intensity is about 0.8 in panel (b) against approximately  $2.5 \times 10^3$  in panel (c)]. Second, both in panels (b) and (c) the maximal beam intensity is achieved right at the center of the waveguide, although many eigenmodes are excited and, moreover, the maximal portion of the energy is concentrated in a higher mode. To illustrate the last statement, in Table 1 we display the coefficients of the expansion (6) of the input Gaussian beam  $q_0(r) = \sqrt{U_0/\pi} e^{-r^2/2}$  over the eigenmodes  $\phi_{\mathbf{n}}$ :

$$c_{\mathbf{n}} = \sqrt{U_0/(2^{n_1+n_2} n_1! n_2!)} (ig_1)^{n_1} (ig_2)^{n_2} e^{g^2/4}. \quad (11)$$

From Table 1 one can see, that for  $\mathbf{g} = (3/2, 0)$  the most excited mode, i.e. the one having the largest coefficient  $c_{\mathbf{n}}$ , is the mode corresponding to  $n_1 = 1$  (rather than the mode with  $n_1 = 0$ ). With further increase of the gradient  $g_1$  the maximal amplitude is shifted towards higher modes.

The obtained results do not exhaust all possibilities of our setup. In particular, for the input beam with the shape of the first Gauss-Hermite function, the solution (10) yields giant amplification of the split beam, as shown in Fig. 1 (d). Even more sophisticated profiles of the amplified beam can be obtained by variation of the input beam,

i.e. by changing  $\mathbf{n}$ , or by using the incidence angle “rotated” with respect to the gain-loss gradient (i.e. using a different relation between  $\mathbf{g}$  and  $\mathbf{a}$ ).

#### 4. Conclusion

To conclude, we have proposed a physical setup where a giant but controllable, i.e. limited by the amplitude, amplification of a guided beam can be achieved during the propagation. Such amplification occurs due to the interference of the multiple gain guided modes. The latter is possible due to the reality of the propagation constants of all modes, on the one hand, and on the other hand due to the energy transfer among the modes which results in exciting many modes, practically by arbitrary input beam. The phenomenon is determined by the parameters of both the system and the input beam, thus allowing for an efficient managing over the output beam amplitude and, to some degree, over the output beam shape. This last property makes the configurations particularly relevant for the practical purposes whenever it is necessary not only to enhance of an input beam until some given amplitude but also change this limit in the course of the experiment by managing only the input field. We also notice that further studies, including different  $\mathcal{PT}$ -symmetric profiles allowing for the spontaneous symmetry breaking or nonlinearity, which may become relevant at high intensities of the phenomenon are of interest.

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